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A priori bounds for semilinear equations and a new class of critical exponents for Lipschitz domains

P.J. McKenna^a, W. Reichel^{b,*,1}^a *Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA*^b *Institut für Mathematik, RWTH-Aachen, Templergraben 55, D-52062 Aachen, Germany*

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Abstract

A priori bounds for positive, very weak solutions of semilinear elliptic boundary value problems $-\Delta u = f(x, u)$ on a bounded domain $\Omega \subset \mathbb{R}^n$ with $u = 0$ on $\partial\Omega$ are studied, where the nonlinearity $0 \leq f(x, s)$ grows at most like s^p . If Ω is a Lipschitz domain we exhibit two exponents p_* and p^* , which depend on the boundary behavior of the Green function and on the smallest interior opening angle of $\partial\Omega$. We prove that for $1 < p < p_*$ all positive very weak solutions are a priori bounded in L^∞ . For $p > p^*$ we construct a nonlinearity $f(x, s) = a(x)s^p$ together with a positive very weak solution which does not belong to L^∞ . Finally we exhibit a class of domains for which $p_* = p^*$. For such domains we have found a true critical exponent for very weak solutions. In the case of smooth domains $p_* = p^* = \frac{n+1}{n-1}$ is an exponent which is well known from classical work of Brezis, Turner [H. Brezis, R.E.L. Turner, On a class of superlinear elliptic problems, *Comm. Partial Differential Equations* 2 (1977) 601–614] and from recent work of Quittner, Souplet [P. Quittner, Ph. Souplet, A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces, *Arch. Ration. Mech. Anal.* 174 (2004) 49–81].

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* Corresponding author. Current address: Institut für Mathematik, Universität Zürich, Winterthurerstr. 190, CH-8057 Zürich, Switzerland.

E-mail address: reichel@math.unizh.ch (W. Reichel).¹ Partially supported by the Deutsche Forschungsgemeinschaft (DFG).

1. Introduction

In this paper we study a priori bounds for positive solutions of the boundary value problem

$$-\Delta u = f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1)$$

where Ω is a bounded Lipschitz domain in \mathbb{R}^n . In this context a priori bounds are understood as follows: there exists a value $M > 0$ such that $\|u\|_\infty \leq M$ for every solution $u \geq 0$ of (1). As test cases one should have in mind $f(x, s) = a(x)s^p$ and $f(x, s) = C(1 + a(x)s^p)$ for some $p > 1$ and with $0 \leq a \in L^\infty(\Omega)$, $\int_\Omega a \, dx > 0$, $C > 0$.

A priori bounds are intimately related to critical exponents, e.g. the critical Sobolev embedding exponent. This is well known for the classical example

$$-\Delta u = a(x)u^p \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (2)$$

If $p \geq \frac{n+2}{n-2}$, $a \equiv a_0 > 0$ and Ω is star-shaped, Pohožaev [15] showed in 1965 that there are no positive weak $H_0^1(\Omega) \cap L^\infty(\Omega)$ -solutions, which can be understood as the failure of a priori bounds. In 1977 Brezis and Turner [4] discovered a priori bounds for weak $H_0^1(\Omega)$ -solutions of (2) if Ω is smooth, $a(x) \geq a_0 > 0$ and $1 < p < p_{\text{BT}} = \frac{n+1}{n-1}$. For $a \equiv a_0 > 0$ this was improved by Gidas, Spruck [11] in 1981 and de Figueiredo, Lions, Nussbaum [8] in 1982 to the range to $1 < p < \frac{n+2}{n-2}$, which is optimal in view of Pohožaev's result.

In [16] Quittner and Souplet extended the a priori bounds of Brezis and Turner in the same range of exponents $1 < p < p_{\text{BT}}$ to *very weak solutions* of (1). The concept of very weak solutions dates back to Stampacchia [19] and was further studied by Brezis et al. in [5]. A function $u \in L^1(\Omega)$ is called a very weak solution of (1) provided $f(x, u) \cdot \text{dist}(x, \partial\Omega) \in L^1(\Omega)$ and

$$\int_\Omega -u \Delta \phi \, dx = \int_\Omega f(x, u) \phi \, dx \quad \text{for all } \phi \in C^2(\overline{\Omega}) \text{ with } \phi|_{\partial\Omega} = 0.$$

As it turns out, positive very weak solutions are very useful in the study of parabolic blow-up, see Section 6. The main tool of Quittner, Souplet was sharp estimates of the heat semigroup obtained earlier by Fila, Souplet and Weissler [10]. Both in [4] and [16] the existence of a priori bounds for $1 < p < p_{\text{BT}}$ depends in an essential way on the smoothness of the domain Ω , i.e., on the fact that a positive copy ϕ_1 of the first Dirichlet eigenfunction of $-\Delta$ on Ω satisfies

$$K^{-1} \text{dist}(x, \partial\Omega) \leq \phi_1(x) \leq K \text{dist}(x, \partial\Omega) \quad (3)$$

for a suitable constant $K > 0$.

Our first contribution in this paper is a study of the naturally arising question what happens if Ω is a square, a hypercube, a conical piece or more generally a Lipschitz domain where (3) fails. As an answer we can define a generalized Brezis–Turner type exponent p_{BT} for a class of Lipschitz domains including the above list. Here we give a brief description of the generalized exponent p_{BT} . Let $G(x, x_0)$ be the Green function with pole at $x_0 \in \Omega$. Assume Ω is such that $G(x, x_0) \geq \text{const} \cdot \text{dist}(x, \partial\Omega)^\gamma$ and suppose that γ is as small as possible. Then $p_{\text{BT}} = \frac{n+\gamma}{n+\gamma-2}$ and $1 < p < p_{\text{BT}}$ guarantees a priori bounds for very weak solutions of (1), provided the notion of a very weak solution is appropriately modified to suit Lipschitz domains. If Ω is a 2-dimensional square then $p_{\text{BT}} = 2$ as compared to $p_{\text{BT}} = 3$ for smooth 2-dimensional domains.

For n -dimensional hypercubes we find $p_{BT} = \frac{n}{n-1}$ as compared to $p_{BT} = \frac{n+1}{n-1}$ for smooth n -dimensional domains. Apart from the range of exponents our assumptions on the nonlinearity $f(x, s)$ are essentially the same as in Brezis, Turner and Quittner, Souplet. On a side note we mention that for the test cases our results allow coefficients $a(x) \geq 0$ vanishing on some part of Ω as long as $\int_{\Omega} a \, dx > 0$.

Next we describe a second major development in the study of a priori bounds for very weak solutions. Until the very recent paper of Souplet [18], the following two (related) questions were open:

- (i) Is there a genuine difference between weak H_0^1 -solutions and very weak solutions of (1)?
- (ii) Is the range of exponents $1 < p < p_{BT}$ sharp?

The results of Brezis, Turner and Quittner, Souplet show that for $1 < p < p_{BT}$ there is no distinction between classical solutions, weak $H_0^1(\Omega)$ -solutions and very weak solutions of (1). However, examples of unbounded very weak solutions of $-\Delta u = C(u+1)^p$ in $\Omega = B_1(0)$ are known for $p > \frac{n}{n-2}$. They are of the form $u(x) = |x|^{-\alpha} - 1$ with $\alpha = 2/(p-1)$ and C chosen appropriately. This solution is not in $H_0^1(\Omega)$ if $p \in (\frac{n}{n-2}, \frac{n+2}{n-2}]$. Note that $p_{BT} < \frac{n}{n-2}$ so that until recently the questions (i) and (ii) remained open for $p \in (p_{BT}, \frac{n}{n-2}]$. Both were finally resolved in [18], where Souplet showed that for all values of $p > p_{BT}$ it is possible to construct $0 \leq a(x) \in L^\infty(\Omega)$, $\int_{\Omega} a(x) \, dx > 0$ and a very weak solution u of

$$-\Delta u = a(x)u^p, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (4)$$

which is neither in $H_0^1(\Omega)$ nor in $L^\infty(\Omega)$. Souplet's example is also based on heat-kernel estimates and fine properties of the heat semigroup. The work of Brezis, Turner [4], Quittner, Souplet [16] and Souplet [18] can therefore be summarized as follows:

For smooth domains and very weak solutions of (1) the exponent $p_{BT} = \frac{n+1}{n-1}$ is a sharp critical exponent.

Our second contribution in this paper is to show that the generalized Brezis–Turner exponents are also sharp for a class of domains such as squares, hypercubes and certain Lipschitz cones in the following sense: if $p > p_{BT}$ then there exist $0 \leq a(x) \in L^\infty(\Omega)$, $\int_{\Omega} a(x) \, dx > 0$ and a very weak solution u of (4), which is neither in $L^\infty(\Omega)$ nor in $H_0^1(\Omega)$. In the course of our investigations, we found a self-contained proof of Souplet's counter-example which does not rely on heat-kernel estimates but instead uses some simple upper- and lower solution arguments.

This paper is organized as follows. In Section 2 we give the main definitions and theorems of this paper. It also contains a generalization of the notion of a *very weak solution* suitable for Lipschitz domains. Section 3 contains the proof of the a priori bound result of Theorem 6 for $1 < p < p_*$, where p_* is defined through the boundary behavior of the Green function. The method of proof depends upon regularity results for very weak solutions and a bootstrap argument. Our approach to the regularity results is via sharp estimates of the Green function on Lipschitz domains, and thus complements the previous approach via heat kernel estimates. Section 4 provides a self-contained proof of the Souplet-type counter-example of Theorem 12 for $p > p^*$ based on upper and lower solutions. Here p^* is a second exponent defined through the notion of a smallest opening angle at the boundary. We do not know if the two exponents p_* and p^* are always equal. Our third main result of Theorem 13 contains a list of domains for

which $p_* = p^*$. It is proved in Section 5. In Section 6 we explain why in the context of Lipschitz domains we were forced to modify the concept of a very weak solution. Moreover, we give a further $W_0^{1,q}$ -regularity result, which applies e.g. to C^1 -domains and convex domains and extends previously known results. We finish this paper with conclusions and open questions in Section 7.

2. Definitions and main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We denote by $\delta(x)$ the distance function to $\partial\Omega$, i.e.,

$$\delta(x) = \min\{|x - y|, y \in \partial\Omega\}.$$

For a given function $0 \leq a \in L^\infty(\Omega)$ with $\int_\Omega a(x) dx > 0$ we denote by $\lambda_{1,a}$ the first Dirichlet eigenvalue of

$$-\Delta u = \lambda a(x)u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

and by $\phi_{1,a}(x) > 0$ the first eigenfunction. For $a \equiv 1$ we use the standard notation λ_1 and ϕ_1 .

Definition 1. On a given bounded Lipschitz domain Ω , fix a positive harmonic function h with $h = 0$ on $\partial\Omega$ and $h = +\infty$ at $P \in \Omega$ and define the superharmonic function $H(x) = \min\{h(x), 1\}$. Let

$$\gamma_* = \inf\{\gamma > 0 \text{ such that } \exists K = K(\gamma) > 0 \text{ with } H(x) \geq K\delta(x)^\gamma\}.$$

Then define the exponent $p_* = \frac{n+\gamma_*}{n+\gamma_*-2}$.

Remark. The above definition does not depend on the choice of the point $P \in \Omega$. Note also that up to a multiple, h coincides with the Dirichlet Green function $G(x, P)$ with pole at $P \in \Omega$. Furthermore, $H(x) \leq \text{const} \cdot \phi_{1,a}(x)$ whenever $\phi_{1,a}$ is a first Dirichlet eigenfunction of $-\Delta$ with weight a as above.

At this stage it is not clear for which Lipschitz domains such a minimal value γ_* is attained. We note that for smooth domains $\gamma_* = 1$ and that in general $\gamma_* \geq 1$. Hence $1 < p_* \leq \frac{n+1}{n-1}$ with equality for smooth domains. The simplest cases for which the value of γ_* is known explicitly are 2-dimensional rectangles with $\gamma_* = 2$, n -dimensional hypercubes with $\gamma_* = n$ and Lipschitz cones, cf. Lemma 8. In Theorem 13 we will describe a class of Lipschitz domains for which we can explicitly determine the value of γ_* .

Definition 2. For a given bounded Lipschitz domain Ω let $m \in C(\overline{\Omega})$ be positive in Ω and $1 \leq p < \infty$. Let $L_m^p(\Omega) = \{v : \Omega \rightarrow \mathbb{R} \text{ measurable: } \int_\Omega |v|^p m dx < \infty\}$ with the norm $\|v\|_{p,m} = (\int_\Omega |v|^p m dx)^{1/p}$.

Note that $\lim_{p \rightarrow \infty} \|v\|_{p,m} = \|v\|_\infty$. Next we define the concept of a very weak solution, which goes back to Brezis et al. [5]. Here we need to modify the definition from [5] because we work on Lipschitz domains rather than smooth domains. More details on the original definition and why it needs to be modified are given in Section 6. For simplicity we begin with the definition of a very weak solution for a linear problem.

Definition 3. Let Ω be a bounded Lipschitz domain with first Dirichlet eigenfunction ϕ_1 . A function $u \in L^1_{\phi_1}(\Omega)$ is called a very weak solution of $-\Delta u = g$ in Ω , $u = 0$ on $\partial\Omega$, if $g \in L^1_{\phi_1}(\Omega)$ and

$$\int_{\Omega} u \eta \, dx = \int_{\Omega} g (-\Delta)^{-1} \eta \, dx$$

for all measurable functions $\eta: \Omega \rightarrow \mathbb{R}$ with $\|\eta/\phi_1\|_{\infty} < \infty$. Here $(-\Delta)^{-1}: L^2(\Omega) \rightarrow W^{1,2}_0(\Omega)$.

Remark. Note that $|\eta| \leq \text{const} \cdot \phi_1$ implies that $|(-\Delta)^{-1} \eta| \leq \text{const} \cdot \phi_1$ by the maximum principle. Hence $\int_{\Omega} g (-\Delta)^{-1} \eta \, dx$ is well defined for $g \in L^1_{\phi_1}(\Omega)$. Note also that a weak $H^1_0(\Omega)$ solution with $g \in L^1(\Omega)$ is also a very weak solution.

Proposition 4 (Existence and uniqueness). Let Ω be a bounded Lipschitz domain and let $g \in L^1_{\phi_1}(\Omega)$. Then $-\Delta u = g$ in Ω with $u = 0$ on $\partial\Omega$ has a unique very weak solution $u \in L^1_{\phi_1}(\Omega)$.

Proof. By splitting $g = g^+ - g^-$, $g^+(x) = \max\{g(x), 0\}$, $g^-(x) = -\min\{g(x), 0\}$ it suffices to prove the proposition in the case $g \geq 0$. Let $g_k(x) = \min\{g(x), k\}$ for some $k \in \mathbb{N}$. Then $g_k \rightarrow g$ in $L^1_{\phi_1}(\Omega)$. Let $u_k \in W^{1,2}_0(\Omega)$ be the weak solution of $-\Delta u_k = g_k$ in Ω with $u_k = 0$ on $\partial\Omega$. Thus

$$\int_{\Omega} u_k \eta \, dx = \int_{\Omega} \nabla u_k \cdot \nabla ((-\Delta)^{-1} \eta) \, dx = \int_{\Omega} g_k (-\Delta)^{-1} \eta \, dx \quad (5)$$

for all measurable η with $\eta/\phi_1 \in L^{\infty}(\Omega)$. Moreover, u_k is monotone increasing in k . Choosing $\eta = \lambda_1 \phi_1$ we obtain from (5)

$$\int_{\Omega} \lambda_1 (u_k - u_l) \phi_1 \, dx = \int_{\Omega} (g_k - g_l) \phi_1 \, dx.$$

For $k > l$ we have $u_k \geq u_l$ and $g_k \geq g_l$. The sequence $(u_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^1_{\phi_1}(\Omega)$ because $(g_k)_{k \in \mathbb{N}}$ converges in $L^1_{\phi_1}(\Omega)$. Hence $u_k \rightarrow u$ in $L^1_{\phi_1}(\Omega)$. Since $\eta/\phi_1 \in L^{\infty}(\Omega)$ implies $(-\Delta)^{-1} \eta/\phi_1 \in L^{\infty}(\Omega)$ we can take the limit $k \rightarrow \infty$ in (5) and find that u is a very weak solution in the sense of Definition 3. Suppose u, \bar{u} are two very weak solutions. Then $\int_{\Omega} (u - \bar{u}) \eta \, dx = 0$ for all measurable η with $\|\eta/\phi_1\|_{\infty} < \infty$. This implies uniqueness $u = \bar{u}$ a.e. in Ω . \square

Next we give the obvious generalization of Definition 3 to very weak solutions of a nonlinear boundary value problem.

Definition 5. Let Ω be a bounded Lipschitz domain with first Dirichlet eigenfunction ϕ_1 . A function $u \in L^1_{\phi_1}(\Omega)$ is called a very weak solution of (1) if $f(\cdot, u(\cdot)) \in L^1_{\phi_1}(\Omega)$ and

$$\int_{\Omega} u \eta \, dx = \int_{\Omega} f(x, u) (-\Delta)^{-1} \eta \, dx$$

for all measurable functions $\eta: \Omega \rightarrow \mathbb{R}$ with $\|\eta/\phi_1\|_\infty < \infty$. Here $(-\Delta)^{-1}: L^2(\Omega) \rightarrow W_0^{1,2}(\Omega)$.

Theorem 6. Assume Ω is a bounded Lipschitz domain with exponent p_* as in Definition 1. Let $f: \overline{\Omega} \times [0, \infty) \rightarrow [0, \infty)$ be a Carathéodory function and assume that there exists a function $0 \leq a \in L^\infty(\Omega)$ with $\int_\Omega a(x) dx > 0$ such that the following holds:

- (i) $\exists C_1 > 0$ and $p \in (1, p_*)$ such that $0 \leq f(x, s) \leq C_1(1 + s^p)$ for all $(x, s) \in \Omega \times [0, \infty)$,
- (ii) $\exists C_2 > 0$ and $\lambda > \lambda_{1,a}$ such that $f(x, s) \geq -C_2 + \lambda a(x)s$ for all $(x, s) \in \Omega \times [0, \infty)$.

Then there exists a value $M > 0$ such that every non-negative very weak solution u of (1) satisfies $\|u\|_\infty \leq M$. Here M depends only on Ω , p , $a(x)$, C_1 , C_2 , λ .

Consider the following examples:

$$-\Delta u = \lambda u + a(x)u^p \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (6)$$

and

$$-\Delta u = \lambda(1 + a(x)u^p) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (7)$$

with p and a as in Theorem 6. Then Theorem 6 applies to (6): for any finite $\Lambda > 0$ there exists a constant $M = M(p, a(x), \Lambda, \Omega)$ such that $\|u\|_\infty \leq M$ for every non-negative very weak solution u of (6) with $\lambda \in [-\Lambda, \lambda_1]$. Similarly, Theorem 6 applies to (7): for any two values $0 < \Lambda_1 < \Lambda_2 < \infty$ there exists a constant $M = M(p, a(x), \Lambda_1, \Lambda_2, \Omega)$ such that $\|u\|_\infty \leq M$ for every non-negative very weak solution u of (7) with $\lambda \in [\Lambda_1, \Lambda_2]$.

Our next goal is twofold: to show by examples that for certain domains a priori bounds fail to exist when p exceeds a second critical value p^* and to give a class of Lipschitz domains for which $p^* = p_*$. This requires that we restrict our attention to Lipschitz domains which possess a “smallest corner.” It is exactly this smallest corner which determines the second critical value p^* and which allows us to show for some classes of domains that $p_* = p^*$. Since the idea of a “smallest corner” is based on the notion of a cone we first define cones and conical pieces.

Definition 7 (Cones, conical pieces). For $x \in \mathbb{R}^n$ let $(r, \theta) \in [0, \infty) \times \mathbb{S}^{n-1}$ be the spherical coordinates of x abbreviated by $x = (r, \theta)$. If $\omega \subset \mathbb{S}^{n-1}$ is open then

$$C_\omega = \bigcup_{r>0} r\omega = \{x = (r, \theta): r > 0, \theta \in \omega\}$$

is a cone with cross-section ω . The set

$$C_\omega^R = C_\omega \cap B_R(0)$$

is called a conical piece with cross-section ω and radius R (see Fig. 1).

Lemma 8. Let $\omega \subset \mathbb{S}^{n-1}$ be a cross-section and let $(\tilde{\lambda}_1, \tilde{\psi}_1)$ be the first Dirichlet eigenvalue, eigenfunction of the Laplace–Beltrami operator $-\Delta_B$ on ω . Moreover, let $\beta = \sqrt{(\frac{n-2}{2})^2 + \tilde{\lambda}_1}$ and $\gamma = \frac{2-n}{2} + \beta$.

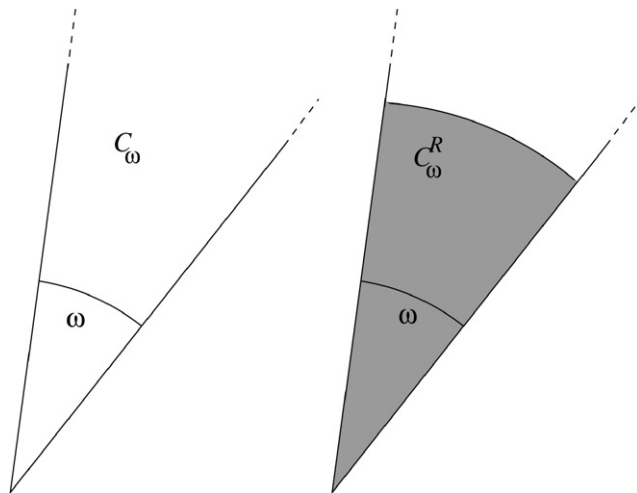


Fig. 1. Cone, conical piece.

(i) The first Dirichlet eigenfunction ϕ_1 of $-\Delta$ on the conical piece C_ω^R is given by

$$\phi_1(x) = \text{const} \cdot J_\beta(\sqrt{\lambda_1}|x|)|x|^{\frac{2-n}{2}}\tilde{\psi}_1(\theta),$$

where J_β is the regular Bessel function with index β , μ is the first zero of J_β on the half-line $[0, \infty)$ and $\lambda_1 = \mu^2/R^2$.

(ii) If ω is a $C^{2,\alpha}$ -smooth cross-section and z is positive, harmonic in C_ω with $z = 0$ on $\partial C_\omega \setminus \{0\}$ and bounded near $x = 0$ then

$$z(x) = \text{const} \cdot |x|^\gamma \tilde{\psi}_1(\theta).$$

Proof. The first Dirichlet eigenfunction ϕ_1 is (up to multiples) the unique eigenfunction of one sign. Therefore, the proof of (i) is a direct computation following from the usual ansatz $\phi_1(x) = r^{\frac{2-n}{2}}\alpha(r)\tilde{\psi}_1(\theta)$ with $r = |x|$. It turns out that $\alpha(r) = J_\beta(\frac{\mu r}{R})$ with $\beta = \sqrt{(\frac{n-2}{2})^2 + \tilde{\lambda}_1}$.

Statement (ii) is more subtle. First we show that $z(x) \rightarrow 0$ as $x \rightarrow 0$. Note that there are two explicit harmonic functions in the cone C_ω vanishing on $\partial C_\omega \setminus \{0\}$:

$$z_\pm(x) = |x|^{\gamma_\pm} \tilde{\psi}(\theta), \quad \gamma_\pm = \frac{2-n}{2} \pm \beta,$$

where $\gamma_+ > 0$ and $\gamma_- < 0$. We may choose $\tau > 0$ so large that $z(x) \leq \tau z_+(x)$ for all $x \in C_\omega$ with $|x| = 1$. This follows from the smoothness of the cross-section and the fact that $\partial_\nu u, \partial_\nu z_+ < 0$ on $\partial C_\omega \setminus \{0\}$. For every $\epsilon > 0$ the maximum principle shows that $0 \leq z(x) \leq \tau z_+(x) + \epsilon z_-(x)$. Letting $\epsilon \rightarrow 0$ we obtain $0 \leq z(x) \leq \tau z_+(x)$ and hence $z(x) \rightarrow 0$ as $x \rightarrow 0$. Now we can apply Theorem 3.3 of Yoshida, Miyamoto [21] to the function $-z$ and obtain the claim (ii) of the lemma. \square

Definition 9 (*Opening angle*). Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $P \in \partial\Omega$. An open Lipschitz domain $\omega \subset \mathbb{S}^{n-1}$ is called the opening angle at P if the following holds: there exist sequences $\sigma_k, \tau_k \subset \mathbb{S}^{n-1}$ of smooth open domains and a sequence of radii $r_k > 0$ such that

- (i) $\bar{\sigma}_1 \subset \bar{\sigma}_2 \subset \cdots \subset \bar{\sigma}_k \subset \omega \subset \bar{\omega} \subset \tau_k \subset \cdots \subset \tau_2 \subset \tau_1$,
- (ii) $\bigcup_{k=1}^{\infty} \sigma_k = \omega, \bar{\omega} = \bigcap_{k=1}^{\infty} \bar{\tau}_k$,
- (iii) $P + C_{\sigma_k}^{r_k} \subset \Omega \cap B_{r_k}(P) \subset P + C_{\tau_k}^{r_k}$.

Definition 10 (*Smallest opening angle*). Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain such that every point $P \in \partial\Omega$ has an opening angle ω_P . Suppose $\sup_{P \in \partial\Omega} \tilde{\lambda}_1(\omega_P) = \tilde{\lambda}_1(\omega_{P_0})$, i.e., the first eigenvalue of the Laplace–Beltrami operator on the opening angle ω_P is maximized at the point P_0 . Then the opening angle ω_{P_0} is called the smallest opening angle of Ω .

For smooth domains the smallest opening angle is the half-sphere \mathbb{S}_+^{n-1} . For planar polygons this definition of an opening angle coincides with the standard notion. The same is true for the following class of domains.

Definition 11. Let Ω be a bounded Lipschitz domain such that $\partial\Omega \setminus \{P_1, \dots, P_K\}$ is smooth and there exists $\rho > 0$ such that for every $i = 1, \dots, K$ the set $\Omega \cap B_\rho(P_i)$ is a conical piece with smooth cross-section ω_i . Then Ω is called a domain with finitely many conical corners.

Based on the definition of a smallest opening angle we can now state the next two results, which will be stated and proved in detail in Sections 4 and 5.

Theorem 12. Let Ω be a bounded Lipschitz domain with smallest opening angle ω_{P_0} and let

$$\gamma^* = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \tilde{\lambda}_1},$$

where $\tilde{\lambda}_1$ is the first Dirichlet eigenvalue of the Laplace–Beltrami operator on ω_{P_0} . If $p > p^* := \frac{n+\gamma^*}{n+\gamma^*-2}$, then there exists a function $0 \leq a \in L^\infty(\Omega)$ with $\int_\Omega a \, dx > 0$ and a positive very weak solution u of

$$-\Delta u = a(x)u^p \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega$$

with $u \notin L^\infty(\Omega)$, $u \notin W_0^{1,2}(\Omega)$. If, moreover, $\frac{n+2}{n-2} > p > p_*$, then there exists a second positive solution $\bar{u} \in W_0^{1,2}(\Omega)$.

Theorem 13. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with smallest opening angle. The following list of domains has the property that the two values γ_* of Definition 1 and γ^* of Theorem 12, and consequently, the two critical exponents p_* and p^* coincide:

- (i) smooth domains,
- (ii) n -dimensional boxes $\Omega = (a_1, b_1) \times \cdots \times (a_n, b_n)$,
- (iii) domains with finitely many conical corners.

All planar polygonal domains are covered by (iii).

3. Proof of Theorem 6

The proof of Theorem 6 is inspired by the recent work of Quittner, Souplet [16] and Dall'Acqua, Sweers [6], see also Bidaut-Véron, Yarus [2] for related results. It is based on the following estimate for the Green function on a bounded Lipschitz domain Ω , which follows from a fundamental result of Bogdan [3]. We note that the results of [3] are stated for $n \geq 3$, but it is clear from the proofs that for $n = 2$ only the nature of the fundamental singularity changes.

Lemma 14. *Let Ω be a bounded Lipschitz domain with Green function $G(x, y)$. Let $h(x) = G(x, P)$ for some $P \in \Omega$ and $H = \min\{h, 1\}$. Suppose $H(x) \geq \text{const} \cdot \delta(x)^\gamma$ and $1 \leq p \leq q \leq \infty$. Then there exists a constant C such that*

$$G(x, y) \leq C|x - y|^{2-n-\gamma(\frac{1}{p}-\frac{1}{q})} H(x)^{-\frac{1}{q}} H(y)^{\frac{1}{p}}, \quad n \geq 3, \quad (8)$$

$$G(x, y) \leq C \log\left(2 + \frac{1}{|x - y|}\right) |x - y|^{-\gamma(\frac{1}{p}-\frac{1}{q})} H(x)^{-\frac{1}{q}} H(y)^{\frac{1}{p}}, \quad n = 2, \quad (9)$$

for all $x, y \in \Omega$.

Proof. One part of the result of Bogdan [3] states the following: suppose for every $Z \in \partial\Omega$ there exists a local boundary parameterization in $B_{r_0}(Z)$ with maximal Lipschitz constant L and let $\kappa = 1/2\sqrt{1+L^2}$. Let furthermore be P as in the lemma and $Q \in \Omega$ such that $|Q - P| = r_0/4$. Then there exists a constant C such that

$$G(x, y) \leq C|x - y|^{2-n} \min\left\{1, \frac{H(x)H(y)}{H(A)^2}\right\}, \quad n \geq 3, \quad (10)$$

$$G(x, y) \leq C \log\left(2 + \frac{1}{|x - y|}\right) \min\left\{1, \frac{H(x)H(y)}{H(A)^2}\right\}, \quad n = 2, \quad (11)$$

for all $x, y \in \Omega$, where A depends on x, y and can be any point with the following properties:

- (i) if $r = \max\{\delta(x), \delta(y), |x - y|\} \leq r_0/32$ then $\delta(A) \geq \kappa r$ and $|x - A|, |y - A| \leq (3 - \kappa)r$,
- (ii) if $r > r_0/32$ then $A = Q$.

It is shown in [3] that such points A exist. For our purposes we only need the following properties of A : there exists $c, C > 0$ such that

$$\text{dist}(A, \partial\Omega) \geq c|x - y|, \quad \frac{H(x)}{H(A)}, \frac{H(y)}{H(A)} \leq C \quad (12)$$

for any $x, y \in \Omega$. The first inequality follows from (i) with $c = \kappa$ if $r \leq r_0/32$. If $r > r_0/32$ then by (ii) we have $\delta(A) \geq c_0 r_0 \geq c_1 \text{diam}(\Omega) \geq c_2 |x - y|$. The second inequality in (12) is shown in the proof of the 3G-theorem, cf. [3, p. 334]. Next we use the following inequality: if $0 \leq s, t \leq K, \sigma \in [-1, 1], \beta \in [0, 1]$ then

$$\min\{1, st\} \leq K^{2\beta} (st)^{1-\beta} \left(\frac{t}{s}\right)^{\beta\sigma}.$$

Applying this to (10) with $s = \frac{H(x)}{H(A)}$, $t = \frac{H(y)}{H(A)}$ and using (12) we obtain for $n \geq 3$

$$\begin{aligned} G(x, y) &\leq C|x - y|^{2-n} \frac{1}{H(A)^{2(1-\beta)}} H(x)^{1-\beta(1+\sigma)} H(y)^{1-\beta(1-\sigma)} \\ &\leq C|x - y|^{2-n-2\gamma(1-\beta)} H(x)^{1-\beta(1+\sigma)} H(y)^{1-\beta(1-\sigma)}. \end{aligned}$$

Finally, with

$$\beta = 1 - \frac{1}{2} \left(\frac{1}{p} - \frac{1}{q} \right), \quad \beta\sigma = \frac{1}{2} \left(\frac{1}{p} + \frac{1}{q} \right)$$

we obtain the result. In this case $0 \leq \beta, \sigma \leq 1$ since $1 \leq p \leq q$. For $n = 2$ we get from (11)

$$\begin{aligned} G(x, y) &\leq C \log \left(2 + \frac{1}{|x - y|} \right) \frac{1}{H(A)^{2(1-\beta)}} H(x)^{1-\beta(1+\sigma)} H(y)^{1-\beta(1-\sigma)} \\ &\leq C \log \left(2 + \frac{1}{|x - y|} \right) |x - y|^{-2\gamma(1-\beta)} H(x)^{1-\beta(1+\sigma)} H(y)^{1-\beta(1-\sigma)}. \end{aligned}$$

The same choice of β and σ as above yields the result. \square

This Green function estimate allows the following regularity result for very weak solutions.

Lemma 15. *Let Ω be a bounded Lipschitz domain with Green function $G(x, y)$. Let $h(x) = G(x, P)$ for some $P \in \Omega$ and $H = \min\{h, 1\}$. Suppose $H(x) \geq \text{const} \cdot \delta(x)^\gamma$. Let $g \in L_H^p(\Omega) \cap L_{\phi_1}^1(\Omega)$ for some $p \geq 1$. Then the very weak solution u of*

$$-\Delta u = g \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

has the following properties:

(i) $u \in L_H^q(\Omega)$ for all $q \in [p, \infty]$ such that

$$\frac{1}{p} - \frac{1}{q} < \frac{2}{n + \gamma}$$

and there exists a constant C such that $\|u\|_{L_H^q} \leq C\|g\|_{L_H^p}$. Equality is admissible except when $p = 1$ or $q = \infty$ or $n = 2$.

(ii) $u \in L^q(\Omega)$ for all $q \in [p, \infty]$ such that

$$\frac{1}{p} - \frac{n}{(n + \gamma)q} < \frac{2}{n + \gamma}$$

and there exists a constant C such that $\|u\|_{L^q} \leq C\|g\|_{L_H^p}$. Equality is admissible except when $p = 1$ or $q = \infty$ or $n = 2$.

Remark. Even in some of the exceptional cases $p = 1$ or $q = \infty$ or $n = 2$ equality may still hold in (ii). Suppose $\partial\Omega \in C^1$ or Ω is convex and let $p > \frac{n+\gamma}{n+1}$. For smooth domains this just means $p > 1$. Then (ii) holds with equality except for $p = \frac{n+\gamma}{n}$ and except for $p = \frac{n+\gamma}{2}, q = \infty$. This follows from Proposition 24 below because in this case $u \in W_0^{1,s}(\Omega)$ with $s = \frac{np}{n+\gamma-p}$ and equality in (ii) follows from the Sobolev embedding theorem (with Orlicz space embedding if $p = \frac{n+\gamma}{2}$, where $q < \infty$ is needed).

Proof. For $g \in L_H^p(\Omega)$ let $\mathcal{G}(g)(x) := \int_{\Omega} G(x, y)g(y) dy$ be the Green operator. We will show the mapping properties of \mathcal{G} corresponding to (i) and (ii) and the norm estimate for \mathcal{G} . Once we have established that \mathcal{G} is a bounded linear operator from $L_H^p(\Omega)$ to $L_H^q(\Omega)$, $L^q(\Omega)$ with the above restrictions on p, q then the regularity results for u follow if we know that the very weak solution u of $-\Delta u = g$ can be represented by convolution of g with the Green function. This can be seen by the following argument: take without loss of generality $g \geq 0$, $g_k = \min\{g, k\}$. Then $u_k = (-\Delta)^{-1}g_k$ can be represented by the Green operator and taking limits $k \rightarrow \infty$ one has $g_k \rightarrow g$, $u_k \rightarrow u$ in $L_{\phi_1}^1(\Omega)$ and hence in $L_H^1(\Omega)$. Moreover, $\mathcal{G}g_k \rightarrow \mathcal{G}g$ in $L_H^1(\Omega)$ and thus $\mathcal{G}g = u$.

The proof of the mapping properties of \mathcal{G} uses the following well-known potential estimate, cf. Gilbarg, Trudinger [12]: for $\alpha \in [2-n, 2)$ consider the Riesz potential operator

$$(Vg)(x) := \int_{\Omega} |x-y|^{2-n-\alpha} g(y) dy.$$

Then V is continuous from $L^p(\Omega)$ into $L^q(\Omega)$ provided $\frac{1}{p} - \frac{1}{q} < \frac{2-\alpha}{n}$. Let us first consider the case $n \geq 3$. By the estimate in Lemma 14 we get

$$|\mathcal{G}(g)(x)| H(x)^{1/q} \leq C \int_{\Omega} |x-y|^{2-n-\gamma(1/p-1/q)} |g(y)| H(y)^{1/p} dy. \quad (13)$$

Using the mapping properties of the Riesz-potential operator we find

$$\|\mathcal{G}g\|_{L_H^q} \leq C \|g\|_{L_H^p}$$

provided $\frac{1}{p} - \frac{1}{q} < (2 - \gamma(\frac{1}{p} - \frac{1}{q}))/n$. This amounts to (i). The equality cases except for $p = 1$, $q = \infty$ follow since under these conditions the mapping properties of the Riesz-potential operator still hold, see Stein [20]. For (ii) we use (13) with $q = \infty$ and obtain

$$\|\mathcal{G}g\|_{L^q} \leq C \|g\|_{L_H^p}$$

provided $\frac{1}{p} - \frac{1}{q} < (2 - \frac{\gamma}{p})/n$. This amounts to (ii). The equality cases in (ii) follow as before.

Finally, in the case $n = 2$ we use the fact that for every $\epsilon \in (0, 1]$ there exists $C = C(\epsilon, \text{diam } \Omega) > 0$ such that $\log(2 + 1/|x-y|) \leq C|x-y|^{-\epsilon}$. With this estimate the proofs of (i) and (ii) work as before, but equality cases can no longer be covered. \square

With this regularity result we can argue like in [16].

Proof of Theorem 6. Recall that the first eigenfunction $\phi_{1,a}$ satisfies $-\Delta\phi_{1,a} = \lambda_{1,a}a(x)\phi_{1,a}$ in Ω with $\phi_{1,a} = 0$ on $\partial\Omega$. For simplicity we write ϕ instead of $\phi_{1,a}$ and assume $\int_{\Omega} \phi \, dx = 1$. Note that $\phi \geq \text{const} \cdot h$ in $\Omega \setminus B_{\epsilon}(P)$ by the maximum principle, where h is the harmonic function of Definition 1 with pole at P . Now let $\gamma > \gamma_*$ and choose a constant $K = K(\gamma)$ such that $H(x) \geq K\delta(x)^{\gamma}$. Using $\eta = \lambda_{1,a}a(x)\phi$ in the definition of a very weak solution and using the hypothesis (ii) we get

$$\lambda_{1,a} \int_{\Omega} a(x)u\phi \, dx = \int_{\Omega} f(x,u)\phi \, dx \geq -C_2 + \lambda \int_{\Omega} a(x)u\phi \, dx.$$

Since $\lambda > \lambda_{1,a}$ we obtain

$$\int_{\Omega} a(x)u\phi \, dx \leq C \quad \text{and} \quad \int_{\Omega} f(x,u)\phi \, dx \leq C, \quad (14)$$

where the bound C is uniform for all non-negative solutions u of (1). Now we can use Lemma 15 with $g(x) = f(x,u) \in L^1_{\phi}(\Omega) \subset L^1_H(\Omega)$ and obtain

$$\|u\|_{L^k_H} \leq C \quad \text{for all } k \in \left[1, \frac{n+\gamma}{n+\gamma-2}\right).$$

Notice that we can choose γ so close to γ_* that we may assume $p < k < \frac{n+\gamma}{n+\gamma-2}$. Choosing k close to $\frac{n+\gamma}{n+\gamma-2}$ we obtain $(p-1)\frac{1}{k} < \frac{2}{n+\gamma}$ and in fact

$$\left(p - \frac{1}{\sigma}\right)\frac{1}{k} < \frac{2}{n+\gamma}$$

for some fixed $\sigma = \sigma(p, k, n, \gamma) > 1$. Next we use a bootstrap argument. Assume that

$$\|u\|_{L^l_H} \leq C \quad \text{for some } l \geq k \quad (15)$$

and set $\tilde{l} = l\sigma$. Note that $\frac{p}{\tilde{l}} - \frac{1}{\tilde{l}} = (p - \frac{1}{\sigma})\frac{1}{l} \leq (p - \frac{1}{\sigma})\frac{1}{k} < \frac{2}{n+\gamma}$. Moreover,

$$\|f(x,u)\|_{L^{l/p}_H} \leq C(1 + \|u\|_{L^l_H}^p) \leq C$$

by (15) and thus by Lemma 15 we obtain $\|u\|_{L^{\tilde{l}}_H} \leq C$, i.e., we have improved (15) from l to \tilde{l} .

After finitely many iterations we have achieved $\|u\|_{L^r_H} \leq C$ with $r > \frac{n+\gamma}{2}$. Hence we may apply Lemma 15 one more time with $q = \infty$. This finishes the proof of Theorem 6. \square

4. Proof of Theorem 12

The proof of Theorem 12 consists in constructing a particular example of a very weak solution of

$$-\Delta u = a(x)u^p \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (16)$$

where $0 \leq a \in L^{\infty}(\Omega)$ and $u \notin L^{\infty}(\Omega)$.

Proof of Theorem 12. Let $\omega = \omega_{P_0} \subset \mathbb{S}^{n-1}$ be the smallest opening angle of Ω and suppose it is attained at $P_0 = 0 \in \partial\Omega$. Let $\alpha \in (n-2, n-2+\gamma^*)$ be fixed such that $\alpha+2 \leq \alpha p$, which requires $p > p^*$ as assumed.

By the definition of the smallest opening angle there exist a smooth cross-section $\sigma \subset \omega$ and a radius $R > 0$ such that the conical piece $\mathcal{C} := C_\sigma^R$ is contained in Ω . Associated to this cross-section σ are the values

$$\bar{\gamma} = \frac{2-n}{2} + \sqrt{\left(\frac{n+2}{2}\right)^2 + \tilde{\lambda}_1^\sigma} \quad \text{and} \quad \bar{p} := \frac{n+\bar{\gamma}}{n+\bar{\gamma}-2}.$$

For any prescribed value $\epsilon > 0$ we may choose the cross-section σ such that $\tilde{\lambda}_1^\omega < \tilde{\lambda}_1^\sigma < \tilde{\lambda}_1^\omega + \epsilon$. In turn, ϵ is assumed to be so small that

$$p > \bar{p} > p^* \quad \text{and} \quad \alpha \in (n-2, n-2+\bar{\gamma}).$$

Let u be the very weak solution of

$$-\Delta u = \frac{1}{|x|^{\alpha+2}} \cdot 1_{\mathcal{C}} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (17)$$

which exists because $\frac{1}{|x|^{\alpha+2}} \in L_{\phi_1}^1(\Omega)$ as shown in Lemma 16. In Lemma 20 (with the help of Lemmas 18 and 19) we will prove that

$$u \geq \text{const} \cdot \frac{1}{|x|^\alpha} \cdot 1_{\mathcal{C}}. \quad (18)$$

Once this is accomplished define

$$a(x) := \frac{1_{\mathcal{C}}}{|x|^{\alpha+2} u(x)^p}$$

and observe that $a(x) \leq |x|^{p\alpha-\alpha-2} \cdot 1_{\mathcal{C}}$, i.e., $0 \leq a \in L^\infty(\Omega)$ due to the choice of α . Moreover, u is a very weak solution of $-\Delta u = a(x)u^p$ in Ω with $u = 0$ on $\partial\Omega$ and clearly $u \notin L^\infty(\Omega)$ and also $u \notin W_0^{1,2}(\Omega)$. \square

The remaining parts of this section consist of five elementary lemmas, which lead towards the key estimate (18).

Lemma 16. Let Ω be a bounded Lipschitz domain with smallest opening angle ω attained at $0 \in \partial\Omega$ and let γ^* be as in Theorem 12. Then $1/|x|^{\alpha+2} \in L_{\phi_1}^1(\Omega)$ for all $\alpha < n-2+\gamma^*$.

Proof. By the definition of the smallest opening angle there is a smooth cross-section $\tau \subset \mathbb{S}^{n-1}$ with $\tau \supset \omega$ and a radius $\tilde{R} > 0$ such that the conical piece $\mathcal{C} := C_\tau^{\tilde{R}}$ contains $\Omega \cap B_{\tilde{R}}(0)$. Associated to the cross-section τ is the value

$$\hat{\gamma} = \frac{2-n}{2} + \sqrt{\left(\frac{n+2}{2}\right)^2 + \tilde{\lambda}_1^\tau}.$$

We may choose τ so close to ω that $\alpha < n - 2 + \hat{\gamma}$. Moreover, by shortening \tilde{R} we may suppose that $\lambda_1(C) > \lambda_1(\Omega)$. Hence, by the maximum principle there is a constant $t > 0$ such that $t\phi_1(C) \geq \phi_1(\Omega)$ in $\Omega \cap B_{\tilde{R}/2}(0)$. Since $\phi_1(C) = \text{const} \cdot |x|^{\hat{\gamma}} \tilde{\psi}_1^{\tau}(\theta)(1 + o(|x|))$ by Lemma 8 we find $\phi_1(\Omega) \leq \text{const} \cdot |x|^{\hat{\gamma}}$ and hence $1/|x|^{\alpha+2} \in L^1_{\phi_1}(\Omega)$. \square

Lemma 17. Let $\mathcal{S} = C_{\sigma'}^R$ be a conical piece with a smooth cross-section $\sigma' \subset \mathbb{R}^{n-1}$. Let ϕ_1 be the first eigenfunction of $-\Delta$ on \mathcal{S} with Dirichlet boundary values. For given $\eta: \mathcal{S} \rightarrow \mathbb{R}$ with $\eta \geq 0$ and $\|\eta/\phi_1\|_{\infty} < \infty$ let $\psi \in W_0^{1,2}(\mathcal{S})$ be the weak solution of $-\Delta\psi = \eta$ in \mathcal{S} with $\psi = 0$ on $\partial\mathcal{S}$. Then there exists a constant c_0 such that the following relations hold as $r = |x| \rightarrow 0$ uniformly for $\theta \in \sigma'$:

$$\begin{aligned}\psi(x) &= c_0 |x|^{\hat{\gamma}} \tilde{\psi}_1(\theta)(1 + o(|x|)), \\ \partial_r \psi(x) &= c_0 \hat{\gamma} |x|^{\hat{\gamma}-1} \tilde{\psi}_1(\theta)(1 + o(|x|))\end{aligned}$$

with $\hat{\gamma} = \frac{2-n}{2} + \sqrt{(\frac{n-2}{2})^2 + \tilde{\lambda}_1}$ and $(\tilde{\lambda}_1, \tilde{\psi}_1)$ the first Dirichlet eigenvalue, eigenfunction of the Laplace–Beltrami operator $-\Delta_B$ on σ' .

Proof. Step 1. First we show that

$$\lim_{x \rightarrow 0} \frac{\psi(x)}{|x|^{\hat{\gamma}} \tilde{\psi}_1(\theta)} = c_0 \quad (19)$$

exists. To see this it is enough to show that $\lim_{x \rightarrow 0} \frac{\psi(x)}{H(x)}$ exists, since $H(x) \approx |x|^{\hat{\gamma}} \tilde{\psi}_1(\theta)$ near $x = 0$, cf. Lemma 21. By the Green function representation of ψ we see that

$$\lim_{x \rightarrow 0} \frac{\psi(x)}{H(x)} = \lim_{x \rightarrow 0} \int_{\Omega} \frac{G(x, y)}{H(x)} \eta(y) dy. \quad (20)$$

The claim would follow, if we could take the limit under the integral in (20), since then $c_0 = \int_{\Omega} K(0, y) \eta(y) dy$, where the function $K(\bar{x}, y) = \lim_{x \rightarrow \bar{x} \in \partial\Omega} G(x, y)/H(x)$ is the Martin kernel. Using (10), (11) and the fact that $|\eta(y)| \leq C\phi_1(y)$ we find

$$\frac{G(x, y)}{H(x)} |\eta(y)| \leq C|x - y|^{2-n} \frac{\phi_1(y)H(y)}{H(A)^2}$$

for $n \geq 3$ and a similar estimate for $n = 2$. For the definition of A see the proof of Lemma 14. Recall that x is close to 0. If y is close to 0 we may take $A = y$ and use that $\phi_1(y)/H(y)$ is bounded near 0. If y is far from 0, then we may take $A = Q$. In both cases we get for small enough x that $\frac{G(x, y)}{H(x)} |\eta(y)| \leq C|x - y|^{2-n}$, and hence the limit $x \rightarrow 0$ may be taken under the integral in (20).

Step 2. Suppose ϕ_1 is normalized by $\phi_1(x) = |x|^{\hat{\gamma}} \tilde{\psi}_1(\theta)(1 + o(|x|))$ as $x \rightarrow 0$. Note that both η and ψ satisfy $0 \leq \eta, \psi \leq C\phi_1$ in \mathcal{S} . Let $\psi_{\lambda}(y) = \lambda^{\hat{\gamma}} \psi(y/\lambda)$ and $\eta_{\lambda}(y) = \lambda^{\hat{\gamma}-2} \eta(y/\lambda)$ for $y \in \lambda\mathcal{S}$. Then $-\Delta\psi_{\lambda} = \eta_{\lambda}$ in $\lambda\mathcal{S}$. Note that $\lambda\mathcal{S}$ exhausts the cone $C_{\sigma'}$ as $\lambda \rightarrow \infty$. Due to $0 \leq \eta, \psi \leq C\phi_1$ and Lemma 8(i) we find on compact subsets of $\bar{C}_{\sigma'}$ that $\eta_{\lambda} \rightarrow 0$ uniformly as $\lambda \rightarrow \infty$ and that ψ_{λ} is uniformly bounded. Using the $W^{2,p}$ -estimates on compact smooth

subsets of $\bar{C}_{\sigma'}$, cf. Gilbarg, Trudinger [12, Theorem 9.13], we find that $\psi_{\lambda_k} \rightarrow z$ in $C^{1,\alpha}$ on compact smooth subsets of $\bar{C}_{\sigma'}$ for a sequence $\lambda_k \rightarrow \infty$. Here z is a non-negative harmonic function with 0 boundary data on $\partial C_{\sigma'} \setminus \{0\}$ which is bounded near $y = 0$. Thus by Lemma 8(ii) we know that $z(y) = d_0|y|^{\hat{\gamma}}\tilde{\psi}_1(\theta)$ for some $d_0 \geq 0$. Choosing a compact smooth subset of $\bar{C}_{\sigma'}$ containing the cross-section σ' we find for $y = (1, \theta)$

$$\lambda^{\hat{\gamma}}\psi(y/\lambda) \rightarrow d_0|y|^{\hat{\gamma}}\tilde{\psi}_1(\theta), \quad \lambda^{\hat{\gamma}-1}(\partial_r\psi)(y/\lambda) \rightarrow \hat{\gamma}d_0|y|^{\hat{\gamma}-1}\tilde{\psi}_1(\theta) \quad (21)$$

for some sequence $\lambda = \lambda_k \rightarrow \infty$ and uniformly for $\theta \in \sigma'$. Due to step 1 the value d_0 coincides with c_0 from (19). This shows that $\psi_\lambda \rightarrow z$ for every sequence $\lambda \rightarrow \infty$ and thus (21) holds for every sequence $\lambda \rightarrow \infty$. Setting $\lambda = 1/|x|$ and $y = x/|x|$ the relation (21) implies the claim. \square

Lemma 18 (Comparison principle). *Let Ω be a bounded Lipschitz domain and let $g \in L^1_{\phi_1}(\Omega)$. Suppose $\underline{w}, \bar{w} \in L^1_{\phi_1}(\Omega)$ satisfy*

$$\int_{\Omega} \underline{w}\eta \, dx \leq \int_{\Omega} g(-\Delta)^{-1}\eta \, dx, \quad \int_{\Omega} \bar{w}\eta \, dx \geq \int_{\Omega} g(-\Delta)^{-1}\eta \, dx,$$

for all measurable non-negative functions $\eta: \Omega \rightarrow \mathbb{R}$ with $\|\eta/\phi_1\|_{\infty} < \infty$. Then $\underline{w} \leq \bar{w}$ a.e. in Ω .

Remark. The functions \underline{w}, \bar{w} are called very weak sub-, supersolution to the problem $-\Delta u = g$ in Ω , $u = 0$ on $\partial\Omega$.

Proof. The conclusion follows from $\int_{\Omega} (\bar{w} - \underline{w})\eta \, dx \geq 0$ for all non-negative η with $\eta/\phi_1 \in L^{\infty}(\Omega)$. Hence $\bar{w} - \underline{w} \geq 0$ a.e. in Ω . \square

Lemma 19. *Let $C = C^R_{\sigma}$ be a conical piece with smooth cross-section $\sigma \subset \mathbb{S}^{n-1}$ and let*

$$\bar{\gamma} = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \tilde{\lambda}_1(\sigma)}.$$

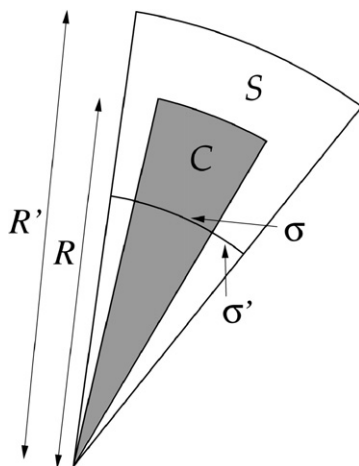
Moreover, let $\alpha \in (n-2, n + \bar{\gamma} - 2)$. Then there exists a second conical piece $S = C^{R'}_{\sigma'} \supset C$ and a very weak subsolution $z: S \rightarrow \mathbb{R}$ satisfying

$$-\Delta z \leq \frac{1}{|x|^{\alpha+2}} \cdot 1_C \quad \text{in } S, \quad z = 0 \quad \text{on } \partial S, \quad (22)$$

such that $z(x) \geq c|x|^{-\alpha} \cdot 1_C$ in S for some positive constant $c > 0$.

Proof. Choose a cross-section $\sigma' \subset \mathbb{S}^{n-1}$ such that $\sigma \Subset \sigma'$ and $\alpha \in (n-2, n + \hat{\gamma})$, where

$$\hat{\gamma} = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \tilde{\lambda}_1(\sigma')}$$

Fig. 2. Construction of S, C .

(see Fig. 2). Define the C^2 -function $z(r, \theta) = (|x|^{-\alpha} - R'^{-\alpha})t\zeta(\theta)$ where $0 < t \leq 1$ and $\zeta : \sigma' \rightarrow (0, \infty)$ is the first eigenfunction of the following problem:

$$-\Delta_B \zeta = \mu_1 \cdot 1_{\sigma} \zeta \quad \text{in } \sigma', \quad \zeta = 0 \quad \text{on } \partial\sigma'.$$

A computation shows

$$\Delta z = (\alpha(\alpha + 2 - n) - (1 - R'^{-\alpha}|x|^{\alpha})1_{\sigma}\mu_1)|x|^{-\alpha-2}t\zeta \quad \text{in } S. \quad (23)$$

We claim that the right-hand side of (23) is larger than $-1_C|x|^{-\alpha-2}$. This is equivalent to the claim that

$$(\alpha(\alpha + 2 - n) - (1 - R'^{-\alpha}|x|^{\alpha})1_{\sigma}\mu_1)t\zeta \geq -1_C \quad \text{in } S. \quad (24)$$

Clearly (24) is true for $x = (r, \theta)$ if $\theta \notin \sigma$. If $\theta \in \sigma$ and $R < r < R'$ then we may choose the radius R' so close to R that $(1 - (R/R')^{\alpha})\mu_1 < \alpha(\alpha + 2 - n)$ and (24) is true also in this case. The remaining case is $x \in C$. In this case we can obtain the bound in (24) by choosing the multiple t sufficiently small.

In this way we obtain a classical (but not yet very weak) subsolution z to (22) on $S_{\epsilon} := S \setminus B_{\epsilon}(0)$ for every $\epsilon > 0$. Also z satisfies $z(x) \geq c|x|^{-\alpha} \cdot 1_C$ for a suitable constant $c > 0$. It remains to verify that z is a very weak subsolution to (22) on S . Next we use the fact that z is a classical and hence weak subsolution, i.e., for every $0 \leq \psi \in W_0^1(S)$

$$\int_{S_{\epsilon}} \nabla z \cdot \nabla \psi \, dx + \oint_{\epsilon \sigma'} \psi \partial_r z \, ds \leq \int_{S_{\epsilon}} \frac{1}{|x|^{\alpha+2}} \cdot 1_C \psi \, dx.$$

For $0 \leq \eta$ with $\eta/\phi_1 \in L^\infty(\mathcal{S})$ let $\psi := (-\Delta)^{-1}\eta$, i.e. $\psi \in W_0^{1,2}(\mathcal{S})$ solves $-\Delta\psi = \eta$ in \mathcal{S} . Moreover, since $\psi \in C^2(\overline{\mathcal{S}_\epsilon})$ we obtain

$$\int_{\mathcal{S}_\epsilon} z\eta \, dx + \oint_{\epsilon\sigma'} (\psi \partial_r z - z \partial_r \psi) \, ds \leq \int_{\mathcal{S}_\epsilon} \frac{1}{|x|^{\alpha+2}} \cdot 1_C \psi \, dx. \quad (25)$$

By the explicit form of z and the fact that $0 \leq \psi \leq \text{const} \cdot \phi_1$ the first boundary integral $\oint_{\epsilon\sigma'} \psi \partial_r z \, dx$ can be estimated by $\text{const} \cdot \epsilon^{n+\hat{\gamma}-2-\alpha} \rightarrow 0$ as $\epsilon \rightarrow 0$. The second boundary integral $\oint_{\epsilon\sigma'} z \partial_r \psi \, dx$ can be estimated similarly by using the explicit form of z and Lemma 17. If we note that $|x|^{-\alpha-2} \in L_{\phi_1}^1(\mathcal{S})$ and also $z \in L_{\phi_1}^1(\mathcal{S})$ we can take the limit $\epsilon \rightarrow 0$ in the remaining integrals in (25) by the monotone convergence theorem. This implies the claim. \square

Lemma 20. *The very weak solution u of (17) is a very weak supersolution to (22) and hence $u \geq z \geq \text{const} \cdot \frac{1}{|x|^\alpha} \cdot 1_C$, where z is the very weak subsolution from Lemma 19.*

Proof. The very weak solution u of (17) may be restricted to \mathcal{S} . We claim that this restriction is a very weak supersolution to (22). To see this let $f := \frac{1}{|x|^{\alpha+2}} \cdot 1_C$ be the right-hand side of (17) and let $f_k := \min\{f, k\}$ for $k \in \mathbb{N}$. Let u_k be the weak $W_0^{1,2}(\Omega)$ -solution of

$$-\Delta u_k = f_k \quad \text{in } \Omega, \quad u_k = 0 \quad \text{on } \partial\Omega. \quad (26)$$

Let $\eta: \mathcal{S} \rightarrow \mathbb{R}$ be a non-negative function such that $\eta/\phi_1^S \in L^\infty(\mathcal{S})$. If ψ is the weak $W_0^{1,2}(\mathcal{S})$ -solution of $-\Delta\psi = \eta$ in \mathcal{S} , $\psi = 0$ on $\partial\mathcal{S}$ let us assume that ψ is extended by zero outside \mathcal{S} . Thus ψ may be used as a test function for (26) which results in

$$\int_{\mathcal{S}} \nabla u_k \cdot \nabla \psi \, dx = \int_{\mathcal{S}} f_k \psi \, dx. \quad (27)$$

Because of the equation solved by ψ and by the fact that $u_k \in W^{1,2}(\Omega)$ we can apply Lemma A.1 of Appendix A and get

$$\int_{\mathcal{S}} \nabla u_k \cdot \nabla \psi \, dx \leq \int_{\mathcal{S}} u_k \eta \, dx. \quad (28)$$

Combining (27) and (28) one finds $\int_{\mathcal{S}} u_k \eta \, dx \geq \int_{\mathcal{S}} f_k \psi \, dx$, where one can pass to the limit $k \rightarrow \infty$. Thus u is a very weak supersolution to (22) on \mathcal{S} . The comparison principle of Lemma 18 shows that $u \geq z$. \square

5. Proof of Theorem 13

Proof. For smooth domains $\gamma_* = 1$ and likewise $\gamma^* = \frac{2-n}{2} + \sqrt{(\frac{n-2}{2})^2 + \tilde{\lambda}_1} = 1$ since the smallest opening angle is the half-sphere \mathbb{S}_+^{n-1} and $\tilde{\lambda}_1(\mathbb{S}_+^{n-1}) = n-1$ with eigenfunction $\tilde{\psi}_1(x) = x_n$.

Let Ω be a domain with finitely many conical corners Q_1, \dots, Q_K and let h be a positive harmonic function with singularity at $P \in \Omega$ which vanishes on $\partial\Omega$. Let us pick a point $Q = Q_i$ (we drop the index i for simplicity) with interior angle $\omega = \omega_i$ and let $x = (r, \theta)$ be polar coordinates

with respect to Q . Choose ρ so small that $|P - Q| > \rho$ and that $\Omega \cap B_\rho(Q)$ coincides with a conical piece. Since ω is smooth we have that $\partial_\nu h < 0$ for $|x - Q| = \rho$, $x \in \partial\Omega$ due to the Hopf maximum principle. Hence there exist values $t_1, t_2 > 0$ such that $t_1 \rho^{\gamma^*} \tilde{\psi}_1(\theta) \leq h(x) \leq t_2 \rho^{\gamma^*} \tilde{\psi}_1(\theta)$ for all $x = (\rho, \theta) \in \Omega$, where $\gamma^* = \frac{2-n}{2} + \sqrt{(\frac{n-2}{2})^2 + \tilde{\lambda}_1^\omega}$ and as usual $(\tilde{\psi}_1, \tilde{\lambda}_1)$ are the first eigenfunction, eigenvalue of $-\Delta_B$ on ω . The maximum principle implies that

$$t_1 |x - Q|^{\gamma^*} \tilde{\psi}_1(\theta) \leq h(x) \leq t_2 |x - Q|^{\gamma^*} \tilde{\psi}_1(\theta) \quad \text{in } \Omega \cap B_\rho(Q) \quad (29)$$

since the upper and lower bounds on h are also harmonic functions. Locally near Q one has $\text{dist}(x, \partial\Omega) = |x - Q| \text{dist}(\theta, \partial\omega)$ and due the smoothness of the cross-section $\text{dist}(\theta, \partial\omega) \approx \tilde{\psi}_1(\theta)$. Hence it follows from (29) that locally near Q the best lower bound for h is given by

$$\begin{aligned} h(x) &\geq \text{const} \cdot |x - Q|^{\gamma^*} \tilde{\psi}_1(\theta) \\ &\geq \text{const} \cdot (|x - Q| \tilde{\psi}_1(\theta))^{\gamma^*} \\ &\geq \text{const} \cdot \text{dist}(x, \partial\Omega)^{\gamma^*} \end{aligned}$$

and the power γ^* cannot be decreased. The optimal lower bound for h in all of Ω is found by maximizing γ^* over all conical corners Q_1, \dots, Q_K . This shows that γ_* from Definition 1 coincides with γ^* from Theorem 12.

For n -dimensional boxes the cross-section is at every corner point isometric to $\mathbb{S}^{n-1} \cap \{x_i > 0: i = 1, \dots, n\}$ and the eigenfunction is $\tilde{\psi}_1(x) = x_1 \cdots x_n$ with eigenvalue $\tilde{\lambda}_1 = 2n(n-1)$ which implies that $\gamma^* = n$. It remains to compute γ_* . The reasoning for the previous domain class is not available since it is based on the smoothness of the cross-section at the corner point and the Hopf maximum principle. However, the following lemma states that positive harmonic functions near a conical point Q satisfy

$$h(x) = \text{const} \cdot |x - Q|^{\gamma^*} \tilde{\psi}_1(\theta) (1 + o(|x - Q|)) \quad \text{as } x \rightarrow Q.$$

With this replacement of (29) the rest of the proof is the same as for the previous domain class. \square

Lemma 21. Let C_ω^R be a conical piece with cross-section $\omega \subset \mathbb{S}^{n-1}$ and let $h: C_\omega^R \rightarrow [0, \infty)$ be a bounded harmonic function with $h = 0$ on $\partial C_\omega^R \setminus \{0\}$. Let $g(\theta) := h(R, \theta)$ and assume $g \in L^2(\omega)$. Furthermore, let $(\tilde{\psi}_i)_{i \in \mathbb{N}}$ be an $L^2(\omega)$ -complete orthonormal set of eigenfunctions of $-\Delta_B$ on ω with corresponding eigenvalues $\tilde{\lambda}_i$. Then the series-expansion

$$h(x) = \sum_{i=1}^{\infty} (|x|/R)^{\gamma_i} (g, \tilde{\psi}_i)_{L^2} \tilde{\psi}_i(\theta) \quad (30)$$

with $\gamma_i = \frac{2-n}{2} + \sqrt{(\frac{n-2}{2})^2 + \tilde{\lambda}_i}$ converges uniformly for $|x| < R$ and hence $h(x) = (g, \tilde{\psi}_1)_{L^2} \times (|x|/R)^{\gamma_1} \tilde{\psi}_1(\theta) (1 + o(|x|))$.

Proof. The boundedness of h implies that $h(x) \rightarrow 0$ as $x \rightarrow 0$. Hence (30) is the correct L^2 -convergent expansion of h . Standard regularity (Moser iteration) implies that $\|\tilde{\psi}_i\|_\infty \leq$

$C\tilde{\lambda}_i\|\psi_i\|_{H^1(\omega)} = C\tilde{\lambda}_i^{3/2}\|\psi_i\|_{L^2(\omega)} = C\tilde{\lambda}_i^{3/2}$. Therefore the series in (30) is dominated by $\|g\|_{L^2(\omega)}\sum_{i=1}^{\infty}(|x|/R)\sqrt{\tilde{\lambda}_i}\tilde{\lambda}_i^{3/2}$. Weyl's asymptotic formula, cf. Davies [7, Theorem 6.3.1], states that $C_1i^{\frac{2}{n-1}} \leq \tilde{\lambda}_i \leq C_2i^{\frac{2}{n-1}}$ for some constants $0 < C_1 < C_2$. In particular, the multiplicity of the i th eigenvalue is at most C_3i , with $C_3 = (\frac{C_2}{C_1})^{\frac{n-1}{2}} - 1$. Hence, the convergence behavior of the series is the same as $\sum_{i=1}^{\infty}(|x|/R)^i i^{\frac{1}{n-1}} i^{\frac{n+2}{n-1}}$ which converges uniformly for $|x| < R$. \square

6. Further properties of very weak solutions

In this section we give more details on the definition of very weak solutions and of the regularity consequences. Consider the linear boundary value problem

$$-\Delta u = g(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (31)$$

Brezis et al. [5] have given the following definition for very weak solutions on smooth domains.

Definition 22. Let Ω be a bounded $C^{2,\alpha}$ -domain and let $g \in L^1_{\phi_1}(\Omega)$. A function $u \in L^1(\Omega)$ is called a very weak solution of (31) if

$$-\int_{\Omega} u \Delta \psi \, dx = \int_{\Omega} g \psi \, dx \quad \forall \psi \in C^2(\overline{\Omega}) \text{ with } \psi|_{\partial\Omega} = 0. \quad (32)$$

The authors of [5] prove existence and uniqueness of very weak solutions on bounded $C^{2,\alpha}$ -domains. Moreover, their definition was motivated by the study of parabolic blow-up. Indeed assume that u is a positive very weak solution of $-\Delta u = f(x, u)$ in Ω with $u = 0$ on $\partial\Omega$. If $0 \leq v_0 \leq u$ then the solution $v(x, t)$ of the parabolic problem $v_t - \Delta v = f(x, v)$ in $\Omega \times (0, T)$, $v(x, t) = 0$ on $\partial\Omega \times (0, T)$ with $v(x, 0) = v_0 \in L^\infty(\Omega)$ does not blow up in finite time, cf. Brezis et al. [5].

Recall from our Definition 5 of very weak solutions on Lipschitz domains that (32) is replaced by

$$\int_{\Omega} u \eta \, dx = \int_{\Omega} g(-\Delta)^{-1} \eta \, dx$$

for all measurable functions $\eta: \Omega \rightarrow \mathbb{R}$ with $\|\eta/\phi_1\|_\infty < \infty$, where $(-\Delta)^{-1}: L^2(\Omega) \rightarrow W_0^{1,2}(\Omega)$. This may seem unnatural, in particular in view of the fact that for $g \in L^1_{\phi_1}(\Omega)$ we prove in Proposition 4 existence of $u \in L^1_{\phi_1}(\Omega)$ instead of $L^1(\Omega)$ as in [5]. However, there are several reasons why there is no other choice than to modify the definition of [5]. For once, as the next results shows, there are examples of Lipschitz domains where the right-hand side is in $L^1_{\phi_1}(\Omega)$ but the solution fails to be in $L^1(\Omega)$. Yet another reason is the following: for smooth domains a natural test function ψ for (32) is given by $-\Delta\psi = 1$ in Ω , $\psi = 0$ on $\partial\Omega$. It is exactly this test-function which establishes that the very weak solution is in fact an L^1 -function. But for Lipschitz domains with small opening angle this function ψ fails to be in $C^2(\overline{\Omega})$.

Proposition 23. Let Ω be a bounded Lipschitz domain with Green function $G(x, y)$. Let $h(x) = G(x, P)$ for some $P \in \Omega$ and $H = \min\{h, 1\}$. Suppose $H(x) \geq \text{const} \cdot \delta(x)^\gamma$. Let $g \in L^1_{\phi_1}(\Omega)$.

- (i) The very weak solution u of (31) in the sense of Definition 5 belongs to $L^1(\Omega)$ if $\gamma < 2$.
- (ii) For $\gamma > 2$ there are examples of domains Ω and $g \in L^1_{\phi_1}(\Omega)$ such that $u \notin L^1(\Omega)$.
- (iii) If $\partial\Omega$ is $C^{2,\alpha}$ then u is in $L^1(\Omega)$ and it is a very weak solution in the sense of Definition 22.

Proof. (i) follows from the regularity result of Lemma 15, part (ii): $p = 1$ and $q = 1$ only work for $\gamma < 2$.

(ii) Let Ω be a domain with one conical corner with smooth opening angle ω such that

$$\gamma = \frac{2-n}{2} + \sqrt{\left(\frac{n-2}{2}\right)^2 + \tilde{\lambda}_1(\omega)}.$$

Consider the example constructed in the proof of Theorem 12: u is the solution of

$$-\Delta u = \frac{1}{|x|^{\alpha+2}} \cdot 1_C \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

and $\alpha < n - 2 + \gamma$ but as close to $n - 2 + \gamma$ as we wish. Then $u(x) \geq \text{const} \cdot |x|^{-\alpha}$. Therefore, if $\alpha > n$ then $u \notin L^1(\Omega)$. Such an α can be chosen provided $\gamma > 2$.

(iii) If $\partial\Omega$ is $C^{2,\alpha}$ then $\gamma = 1$ and (i) implies $u \in L^1(\Omega)$. Since both Definitions 22 and 5 produce a unique solution, the two concepts coincide in this case. \square

We conclude this section with a further regularity result for very weak solutions on certain Lipschitz domains. Let Ω be a Lipschitz domain. Let $s > 1$ and $1/s + 1/s' = 1$. We say that Ω has property $P(s)$, cf. Simader, Sohr [17], if for every $u \in W_0^{1,s}(\Omega)$ the functional

$$\mathcal{L}: \begin{cases} W_0^{1,s'}(\Omega) \rightarrow \mathbb{R}, \\ \psi \mapsto \int_{\Omega} \nabla u \cdot \nabla \psi \, dx \end{cases}$$

satisfies

$$C \|u\|_{W_0^{1,s}} \leq \|\mathcal{L}\| = \sup \left\{ \int_{\Omega} \nabla u \cdot \nabla \psi: \psi \in W_0^{1,s'}(\Omega), \|\psi\|_{W_0^{1,s'}} = 1 \right\}$$

for a constant $C = C(s, \Omega)$. One can show that the fact that property $P(s)$ holds for every $s \in (1, \infty)$ is equivalent to the solvability in $W_0^{1,p}(\Omega)$ of $-\Delta u = g$ in Ω , $u = 0$ on $\partial\Omega$ with $g \in W^{-1,p}(\Omega)$ for all $p \in (1, \infty)$ (see the proof of Proposition 24 below).

If Ω is a C^1 -domain, property $P(s)$ is shown to hold for all $s > 1$ in Simader, Sohr [17, Chapter II, Theorem 1.1]. If Ω is a convex domain then the same was shown by Alkhutov and Kondrat'ev [1]. This shows that for a non-convex conical piece, i.e. with cross-section $\omega \supset \mathbb{S}_+^n$, property $P(s)$ cannot hold for every $s > 1$. Take, for example, the first eigenfunction which is in $L^\infty(\Omega)$, but certainly not in high $W_0^{1,p}$ -spaces, cf. Lemma 8(i).

Finally, let us mention that

$$\|\mathcal{L}\| = \inf\{\|\nabla u - z\|_{L^s(\Omega)} : z \in L^s(\Omega), \operatorname{div} z = 0\},$$

where $\operatorname{div} z = 0$ is understood in the sense that $\int_{\Omega} z \cdot \nabla \phi \, dx = 0$ for all $\phi \in W_0^{1,s'}(\Omega)$. The infimum in the above characterization of $\|\mathcal{L}\|$ is attained. We do not know if this characterization of $\|\mathcal{L}\|$ could be useful.

In the case of a smooth domain ($\gamma = 1$) the following result is due to [10,18]. We can slightly sharpen their result in the sense that with the exception of $p = \frac{n+1}{n}$ we can allow equality in the relation $\frac{1}{p} - \frac{n}{(n+1)q} \leq \frac{1}{n+1}$.

Proposition 24. *Let Ω be a bounded Lipschitz domain with Green function $G(x, y)$ and suppose Ω has property $P(s)$ for all $s > 1$. Let $h(x) = G(x, P)$ for some $P \in \Omega$ and $H = \min\{h, 1\}$. Suppose $H(x) \geq \operatorname{const} \cdot \delta(x)^\gamma$. Let $g \in L_H^p(\Omega) \cap L_{\phi_1}^1(\Omega)$ for some $p \geq 1$. Then the very weak solution u of*

$$-\Delta u = g \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

satisfies $u \in W_0^{1,q}(\Omega)$ for all $q > 1$ such that $\frac{1}{p} - \frac{n}{(n+\gamma)q} \leq \frac{1}{n+\gamma}$ with the exception of $p = \frac{n+\gamma}{n}$, where strict inequality, i.e. $1 < q < \frac{n}{n-1}$ is required. Moreover, the inequality $\|u\|_{W_0^{1,q}} \leq C \|g\|_{L_H^p}$ holds with a constant C independent of g and u .

Remark. In order to get $u \in W_0^{1,1+\epsilon}(\Omega)$ one needs $p > \frac{n+\gamma}{n+1}$. In this case the solution u has a trace and fulfills the boundary condition pointwise almost everywhere and it fulfills the equation in the sense $\int_{\Omega} \nabla u \nabla \psi \, dx = \int_{\Omega} g \psi \, dx$ for all $\psi \in C_0^\infty(\Omega)$. The solution constructed in Theorem 12 belongs to $W_0^{1,1+\epsilon}(\Omega)$ only for $\gamma = 1$.

The proof of Proposition 24 needs the following lemma and the Hardy–Sobolev inequality. Recall Hardy’s inequality on bounded Lipschitz domains, cf. Opic, Kufner [14]:

$$\int_{\Omega} \frac{|u(x)|^\tau}{\delta(x)^\tau} \, dx \leq \frac{1}{C_H} \int_{\Omega} |\nabla u|^\tau \, dx \quad \text{for all } u \in W_0^{1,\tau}(\Omega)$$

with $C_H = C_H(\Omega)$ and the Hardy–Sobolev inequality

$$\int_{\Omega} \frac{|u(x)|^\alpha}{\delta(x)^\beta} \, dx \leq C_{HS}(\alpha, \beta) \left(\int_{\Omega} |\nabla u|^\tau \, dx \right)^{\alpha/\tau} \quad \text{for all } u \in W_0^{1,\tau}(\Omega)$$

with $\beta \leq \alpha \leq \frac{\tau}{n-\tau}(n-\beta)$, $0 \leq \beta < \tau$ if $n > \tau$ and $\beta \leq \alpha < \infty$, $0 \leq \beta < \tau$ if $n \leq \tau$. For self-containment of this paper let us give a quick proof of the Hardy–Sobolev inequality. First, by Hölder’s inequality

$$\int_{\Omega} \frac{|u|^\alpha}{\delta^\beta} \, dx = \int_{\Omega} \frac{|u|^\beta}{\delta^\beta} |u|^{\alpha-\beta} \, dx \leq \left(\int_{\Omega} \frac{|u|^\tau}{\delta^\tau} \, dx \right)^{\beta/\tau} \left(\int_{\Omega} |u|^{\frac{\tau(\alpha-\beta)}{\tau-\beta}} \, dx \right)^{\frac{\tau-\beta}{\tau}}.$$

The first integral can be estimated by Hardy's inequality and the second by the Sobolev inequality provided $\alpha \geq \beta$, $\tau > \beta$ and $\alpha \leq \frac{\tau(n-\beta)}{n-\tau}$ if $\tau < n$ and $\alpha < \infty$ if $\tau \geq n$.

Lemma 25. Let Ω be a bounded Lipschitz domain with Green function $G(x, y)$. Let $h(x) = G(x, P)$ for some $P \in \Omega$ and $H = \min\{h, 1\}$. Suppose $H(x) \geq \text{const} \cdot \delta(x)^\gamma$. For $p \geq 1$ let $g \in L_H^p(\Omega)$ and let

$$q' \geq \begin{cases} \frac{np}{(n+1)p-n-\gamma} & \text{if } p < n + \gamma, \\ 1 & \text{if } p \geq n + \gamma, \end{cases}$$

with the exception of $p = \frac{n+\gamma}{n}$ where we require $q' > n$. Then the functional

$$l: \begin{cases} W_0^{1,q'}(\Omega) \rightarrow \mathbb{R}, \\ \psi \mapsto \int_\Omega g \psi dx \end{cases}$$

is a bounded linear functional with $\|l\| \leq C \|g\|_{L_H^p}$.

Remark. Note that the restrictions on q' are precisely the restrictions on q in Proposition 24.

Proof. We distinguish two cases: $1 \leq p < \frac{n+\gamma}{n}$ and $p > \frac{n+\gamma}{n}$. The exceptional case $p = \frac{n+\gamma}{n}$ is treated by $g \in L_H^{\tilde{p}}(\Omega)$ for any $\tilde{p} < p$.

Assume $1 \leq p < \frac{n+\gamma}{n}$. Then the value q' from the statement of the lemma is larger than n . Hence we have the embedding $W_0^{1,q'}(\Omega) \rightarrow C_0^{1-n/q'}(\Omega)$ and thus

$$|\psi(x)| \leq C \|\psi\|_{W_0^{1,q'}} \delta(x)^{1-n/q'}.$$

Next we estimate

$$\begin{aligned} \int_\Omega |g\psi| dx &= \int_\Omega |g| H^{1/p} |\psi| H^{-1/p} dx \\ &\leq \|g\|_{L_H^p} \left(\int_\Omega |\psi|^{p/(p-1)} H^{-1/(p-1)} dx \right)^{(p-1)/p} \\ &\leq C \|g\|_{L_H^p} \left(\int_\Omega |\psi|^{p/(p-1)-q'} |\psi|^{q'} \delta^{-\gamma/(p-1)} dx \right)^{(p-1)/p} \\ &\leq C \|g\|_{L_H^p} \|\psi\|_{W_0^{1,q'}}^{1-\frac{q'}{p}(p-1)} \left(\int_\Omega |\psi|^{q'} \delta^s dx \right)^{(p-1)/p}, \end{aligned}$$

where we have used $|\psi(x)| \leq C \|\psi\|_{W_0^{1,q'}} \delta(x)^{1-n/q'}$ and where

$$s = \left(1 - \frac{n}{q'}\right) \left(\frac{p}{p-1} - q'\right) - \frac{\gamma}{p-1},$$

and a short computation shows that $s \geq -q'$. Hence Hardy's inequality may be applied to the last integral in the above chain of inequalities. This leads to $\int_{\Omega} |g\psi| dx \leq C \|g\|_{L_H^p} \|\psi\|_{W_0^{1,q'}}$ which shows the boundedness of the functional l .

Now assume $p > \frac{n+\gamma}{n}$, i.e., $q' < n$. As before we find

$$\int_{\Omega} |g\psi| dx \leq C \|g\|_{L_H^p} \left(\int_{\Omega} |\psi|^{p/(p-1)} \delta^{-\gamma/(p-1)} dx \right)^{(p-1)/p}$$

and now we need to show that the last integral can be estimated by the Hardy–Sobolev inequality. With $\tau = q'$, $\alpha = p/(p-1)$ and $\beta = \gamma/(p-1)$ we check the three conditions: first, we are in the case $\tau < n$. Second, $\beta = \gamma/(p-1)$ is less than $\frac{np}{(n+1)p-n-\gamma} \leq q'$ since the former is decreasing in p , the latter increasing in p and they meet at $p = (n+\gamma)/n$. Hence $\beta < \tau$. Finally one needs to check the inequality $\alpha \leq \frac{\tau(n-\beta)}{n-\tau}$. This is equivalent to $q' \geq \frac{np}{(n+1)p-n-\gamma}$. Hence, as before $\int_{\Omega} |g\psi| dx \leq C \|g\|_{L_H^p} \|\psi\|_{W_0^{1,q'}}$, i.e. the functional l is bounded. \square

Proof of Proposition 24. The proof follows the ideas in Simader, Sohr [17, Chapter II]. The functional l defined in Lemma 25 is a bounded linear functional on $W_0^{1,q'}(\Omega)$. The claim follows if we can show that there exists $u \in W_0^{1,q}(\Omega)$ such that $l(\psi) = \int_{\Omega} \nabla u \cdot \nabla \psi dx$ for all $\psi \in W_0^{1,q'}(\Omega)$. For this purpose define the continuous linear operator

$$Z: \begin{cases} W_0^{1,q}(\Omega) \rightarrow (W_0^{1,q'}(\Omega))^*, \\ u \mapsto Z_u \text{ where } Z_u(\psi) = \int_{\Omega} \nabla u \cdot \nabla \psi dx. \end{cases}$$

The proof is done if Z is onto. We claim that the image $Z(W_0^{1,q}(\Omega))$ is closed. Suppose $Z_{u_k} \rightarrow \mathcal{L} \in (W_0^{1,q'}(\Omega))^*$ as $k \rightarrow \infty$. By property $P(q)$ we know that u_k is a Cauchy sequence and hence converges to $u \in W_0^{1,q}(\Omega)$. The continuity of Z implies that $\mathcal{L} = Z(u)$, i.e., the image space is closed. Suppose for contradiction that Z is not onto. By the Hahn–Banach theorem there exists $F \in (W_0^{1,q'}(\Omega))^{**}$ such that $\|F\| = 1$ and $F(Z_u) = 0$ for all $u \in W_0^{1,q}(\Omega)$. By reflexivity, there is $f \in W_0^{1,q'}(\Omega)$, $\|f\|_{W_0^{1,q'}} = 1$ such that

$$0 = F(Z_u) = Z_u(f) = \int_{\Omega} \nabla u \cdot \nabla f dx \quad \text{for all } u \in W_0^{1,q}(\Omega).$$

By property $P(q')$ we obtain

$$C(q', \Omega) \|f\|_{W_0^{1,q'}} \leq \sup \left\{ \int_{\Omega} \nabla u \cdot \nabla f dx : \|u\|_{W_0^{1,q}} = 1 \right\} = 0,$$

which is a contradiction. \square

7. Conclusions and open questions

We finish our paper with a set of comments and questions, which we could not resolve so far.

Problem 1. For which class of domains is $p_* = p^*$?

In Theorem 13 a list of domains is given for which this is true.

Problem 2. Let $G(x, P)$ be the Green function of Ω with pole at $P \in \Omega$. What is the most general class of domains for which $H(x) = \min\{G(x, P), 1\}$ and $\phi_1(x)$ are comparable?

Recall that on one hand very weak solutions are defined by belonging to $L^1_{\phi_1}(\Omega)$ but on the other hand the regularity gain in Lemma 15 happens in $L^q_H(\Omega)$ and not in $L^q_{\phi_1}(\Omega)$. In the case of smooth domains this distinction does not occur.

Problem 3. Let $0 \leq a \in L^\infty(\Omega)$ with $\int_\Omega a(x) dx > 0$ and consider $-\Delta u = a(x)u^p$ in Ω with $u = 0$ on $\partial\Omega$. If $p > p^*$ is there always a positive, unbounded, very weak solution?

Note that in Theorem 12 only one such example was constructed. The case $a(x) \equiv 1$ alone is very interesting and has recently received a partial answer by del Pino, Musso, Pacard [9]. The authors show that for the above problem with $a \equiv 1$, $\partial\Omega$ smooth there exists $\epsilon > 0$ such that for $p \in [\frac{n+1}{n-1}, \frac{n+1}{n-1} + \epsilon)$ a positive very weak unbounded solution exists. It has a finite but arbitrarily large number of prescribed blow-up points on $\partial\Omega$.

The result of del Pino, Musso and Pacard [9] has also surprising implications on symmetry properties of very weak solutions on symmetric domains. Their result shows in particular that the Gidas–Ni–Nirenberg result cannot hold for unbounded, very weak solutions.

This symmetry question naturally leads to another set of open problems. In a recent paper [13] we considered the question of symmetry of positive solutions of

$$-\Delta_h u = f(u) \quad \text{in } \Omega_h, \quad u = 0 \quad \text{on } \partial\Omega_h, \quad (33)$$

where

$$\Delta_h u(x) = \sum_{j=1}^n \frac{u(x + h_j e_j) - 2u(x) + u(x - h_j e_j)}{h_j^2}$$

is the discretization of the Laplace operator on a n -dimensional hyper-cube $\Omega = (-a_1, a_1) \times \cdots \times (-a_n, a_n)$, $\{e_1, \dots, e_n\}$ is the standard-basis of \mathbb{R}^n and $h = (h_1, \dots, h_n)$ is the mesh-size vector. The following answer to the symmetry question was given: for a certain class of nonlinearities solutions are asymptotically symmetric provided $\|u_h\|_\infty \leq M$ uniformly for all solutions u_h of (33), where the a priori bound is assumed to be uniform in the mesh-size vector h . Also explicit estimates on the defect of symmetry depending on M , f and h were given. The result of del Pino, Musso, Pacard and the failure of the Gidas–Ni–Nirenberg theorem for unbounded very weak solutions suggests that this uniform boundedness assumption is also a necessary condition for asymptotic symmetry. Thus the symmetry problem for solutions of nonlinear finite difference boundary value problems leads immediately to the following open problem.

Problem 4. *Can one prove a priori bounds for positive solutions of the discretized problem (33) in the same range of exponents as for the continuous case?*

Finally we pose the slightly provocative question:

How important are very weak solutions, which are not in $L^\infty(\Omega)$ and not in $H_0^1(\Omega)$?

Obviously numerical approximations to solutions of $-\Delta u = f(x, u)$ in Ω with $u = 0$ on $\partial\Omega$ are important. If very weak solutions of such equations exist then they will show up when one tries to numerically solve such problems. Standard methods for finding solutions of discretized equations such as constrained optimization or mountain-pass algorithms are usually assumed to produce approximations of weak $H_0^1(\Omega)$ -solutions rather than approximations of unbounded very weak solutions. However, such an algorithm might accidentally come close to such a very weak solution and stop. Thus, the following questions naturally arise:

Problem 5. *How can one calculate discrete approximations to very weak solutions, which are not in $H_0^1(\Omega)$ and not in $L^\infty(\Omega)$?*

Problem 6. *How can one distinguish between discrete approximations to $H_0^1(\Omega)$ -solutions and to unbounded very weak solutions?*

Problem 7. *At each discretization level, a finite discrete solution u_h of (33) has a well-defined Morse index. If u_h converges to an unbounded very weak solution as $h \rightarrow 0$, will its Morse index remain bounded or become unbounded?*

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Appendix A

The following lemma was used in the proof of Lemma 20. We give it for the convenience of the reader, although it might be known.

Lemma A.1. *Let D be a bounded Lipschitz domain, $0 \leq \eta \in L^2(D)$ and $0 \leq \psi \in W_0^{1,2}(D)$ a weak solution of*

$$-\Delta \psi = \eta \quad \text{in } D, \quad \psi = 0 \quad \text{on } \partial D. \quad (\text{A.1})$$

Then

$$\int_D \nabla \psi \cdot \nabla \phi \leq \int_D \eta \phi \, dx \quad \forall \phi \in W^{1,2}(D) \text{ with } \phi \geq 0. \quad (\text{A.2})$$

Proof. Let us first prove the result for $\eta \in C^\infty(D)$. Then $\psi \in C^\infty(D)$ and (A.1) holds pointwise in D . By Sard's lemma for almost every $0 < s < \|\psi\|_\infty$ the super-level set $D_s = \{x \in D: \psi(x) > s\}$ has a smooth boundary. Thus we obtain for almost every $s \in (0, \|\psi\|_\infty)$ and every $\phi \in W^{1,2}(D)$ with $\phi \geq 0$

$$\int_{D_s} \nabla \psi \cdot \nabla \phi \, dx = \int_{D_s} \eta \phi \, ds + \oint_{\partial D_s} \underbrace{\phi \, \partial_\nu \psi}_{\leq 0} \, ds \leq \int_{D_s} \eta \phi \, ds.$$

Choosing an appropriate sequence $s \rightarrow 0$ we obtain (A.2). For the general case we can approximate $\eta \in L^2(D)$ by a sequence $\eta_k \in C^\infty(D)$ with $\eta_k \rightarrow \eta$ in $L^2(D)$. Let $\psi_k \in W_0^{1,2}(D) \cap C^\infty(D)$ be the corresponding solution. Then (A.2) holds for ψ_k, η_k and every test function $\phi \in C^\infty(D)$ with $\phi \geq 0$. Letting $k \rightarrow \infty$ we retrieve the result for ψ, η . \square

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